# TEACHING CLASSES IN GEOMETRY TO SOLVE ORIGINAL EXERCISES.*

By Fletcher Durell.

We all realize, I presume, that we have a difficult problem before us. I feel that I shall be more likely to say something that you will care to listen to if I confine myself largely to personal experience. I will however give a little theory at the outset.

It has often been remarked that a pupil in studying geometry is learning to use a certain set of tools, viz.: the line, angle, triangle, circle, etc. In like manner it is stated that in learning to solve original exercises he has a kit or chest of tools before him and is learning to select for himself that geometric instrument or set of instruments which will accomplish a desired result. Thus when he attempts to prove that the bisectors of the opposite angles of a parallelogram are parallel, the pupil is taught to observe that he has on the figure a quadrilateral, two triangles, and various sets of parallel lines, and to recall the various theorems relating to each of these objects or sets of objects, and to test or to try each of these theorems in succession till he finds one or more which will enable him to prove the desired result. In this way it is possible to get several different proofs for the original exercise just mentioned.

But in practice this kit-of-tools process as ordinarily stated is found very fragmentary and defective. It surely is a loose and unsystematic method of procedure to take a mere catalogue of geometric objects and the theorems relating to them and to try these in additive succession till we strike one that will answer our purpose. Also in this process, the tendency is to limit our tests to those specific geometric concepts which are closely related to the matter in hand, such as parallel lines, the triangle, and parallelogram, and neglect more general, and hence on the whole more important and more powerful toolages such as the axioms, and proofs by superposition, or by negative methods. Also the use of auxiliary quantities, the most important and powerful tool in proving difficult originals and the one needing most careful control and the most systematic use, is left wholly to haphazard and accidental treatment.

It has occurred to me that it is important in this connection

* Read at a meeting of the Philadelphia Section.

to raise the question of the nature of a geometric tool. What is the inner essence or common property of all geometric instruments, and indeed of implements used in work of any kind? The answer seems to be that a geometric tool is something external to the data of a problem, just as a saw or axe is external to the mechanic using it. Proof is a step-by-step process. But it cannot be this unless we have something to work with. Without this, instead of a proof we have immediate intuition. An instrument is a supplementary something used in an auxiliary way. Toolage means auxiliary quantity or entity. This latter idea is more general than that of a set of particular tools and includes any number of tools as particular cases of itself. Being general and continuous it should open the way to a more comprehensive classification and a more systematic treatment of originals in geometry.

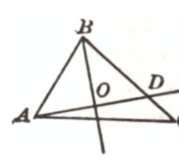
Let me present a specimen way in which I think the above idea can be made a source of progress in this department of mathematical pedagogy.

Geometric toolages, *i. e.:* Auxiliary quantities may be first classified as natural or artificial, that is, as that which is existent or given us in a problem or that which we create and introduce.

An example of a natural or existent auxiliary quantity is the third angle which is used in proving the theorem that two opposite or vertical angles formed by intersecting straight lines are equal. An example of artificial auxiliary quantity is the line drawn to bisect the vertical angle of an isosceles triangle in the proof of the theorem that angles at the base of an isosceles triangle are equal.

Again each of these forms of toolage may be subdivided into internal or external instrumentalism, that is, into auxiliary quantity internal or external in position with reference to the data of a problem. Combining these ideas we have the tools or toolage entities of geometry classified into four primary groups which we shall denote by $A, B, C, D$.

A. *Natural Internal Auxiliary Quantity.*—Thus if on the figure to the left we have given that $ABC$ is any triangle, that $BO$ bisects the angle $ABC$ and $AD$ is perpendicular to $BO$ meeting $BO$ at $O$ and $BC$ at $D$, and it is required to prove $\triangle ABO = \triangle DBO$; and if we prove the tri-



angles equal by means of the sides and angles which are internal to or a part of the triangles, we are using toolage of the species A.

B. *Natural External Auxiliary Quantity.*—An example of B occurs in proving the diagonals of a rectangle equal by means of a pair of triangles of which the diagonals named form a part or detail, the triangles being existent on the given figure but being in the main external to the given diagonals.

The third angle used in proving two opposite or vertical angles equal mentioned above in another illustration of species B

C. *Artificial External Auxiliary Quantity.*—The auxiliary line used in proving the base angles of an isosceles triangle equal also mentioned above, is an ample of C. We have another example in the auxiliary line or lines drawn within an isosceles trapezoid in the process of proving the base angles of an isosceles trapezoid equal. Of course innumerable examples of this as well as of the other species named occur throughout geometry.

D. *Artificial External Auxiliary Quantity.*—A familiar example of the D species is furnished by the auxiliary line or lines drawn outside of a triangle in proving the sum of the angles of a triangle equal to two right angles.

It is also important to make subdivisions of D. Artificial external quantity may be subdivided into forms which are (1) Finite or (2) infinite.

An instance of the finite variety of D is the circumference drawn to circumscribe a regular polygon in proving properties of the polygon. An instance of the infinite variety of D is the use of three-dimensional space when in a proof by superposition a geometric object is picked up out of a plane and turned over and put back in the plane. Also this infinite variety of D may take two main forms, (*a*) the geometric form as in the use of three-dimensional space just referred to, or in proof by use of an unlimited locus, (*b*) the logical infinite such as is used in negative demonstrations as when we prove one object greater than another by proving that it is not equal to it or less than it. Here we obtain our proof by the use of an infinite logical negative matrix.

The differences in the above general A, B, C, D methods of proof find an illustration in the various possible proofs of the theorem that the diagonals of a regular pentagon are equal.

If we prove this property by means of a pair of triangles formed by the diagonals and the sides of a pentagon, we use B; if by dropping perpendiculars from the vertices of the pentagon upon the diagonals, we use C; if we obtain a proof by circumscribing a circle about the given regular pentagon we use D, 1; if by superposition of triangles, we use D, 2.

It should also be observed at this point that the axioms find a natural place in the above schematism. For the axioms are seen to be special and important cases of A, B, C, D. Thus the axiom that a whole is equal to the sum of its parts, falls under A; the axiom that things equal to the same thing are equal to each other, falls under B, or D, etc. Hence, to our A, B, C, D we may add E, the axioms.

It is also recognized, of course, that the above special forms or ways of applying auxiliary objects or entities may be combined and compounded variously.

It may be well to tabulate the results of our analysis. We have:

I. *General Geometric Toolages.*
- A. Natural Internal
- B. Natural External
- C. Artificial Internal
- D. Artificial External
    - 1. Finite
    - 2. Infinite
        - (*a*) Geometric
          Ex. Proof by superposition or by the locus
        - (*b*) Logical
          Ex. Proofs by negative methods
- E. The axioms (= important special cases of A, B, C, D, and are broader than geometry).

II. *Specific Toolages.*
- X. Geometric objects considered
- Y. Known theorems concerning these geometric objects, considered singly or in combination.

I will not at this place try to make a complete statement of the virtues or defects of the above method of classifying the instruments used in geometric proofs. I will, however, point out such of its qualities as now open the way to a further development of the plan.

1. This A, B, C, D method proceeds from the simple to the complex and in particular opens the way to check the pupil and especially the weak pupil, in his wild tendency to use all sorts of auxiliary lines. In order to be able to control this fatal tendency

it is well to note its probable cause. This tendency on the part of the pupil probably arises from the fact that auxiliary quantities of the C and D types are visible and palpable and hence seem to hold out larger promises of help to the pupil while auxiliary objects of the A and B form are often implicit and a matter of relations which do not strike the eye vividly. Hence we realize the prime importance of holding pupils to the A and B methods of proof till these are thoroughly mastered before C and D are taken up.

2. It incorporates the axioms in our system of work, and gives them a standard place therein.

3. It provides for the regular and legitimate use of proof by superposition, and for negative demonstrations. The prevalent dislike for these methods of proof, and the feeling of doubt on the part of many as to their full validity, are probably due to our reluctance to use such a large amount of auxiliary toolage, so large indeed that we feel that we may not have it under complete control. But our scheme instead of this occasional use under protest gives a definite though subordinate position and function to these methods of proof.

4. It also opens the way to the free use of concepts of number, and of the algebraic unknown quantity, as we shall see later.

We will now consider the more definite and immediately practical ways in which the above general considerations may be applied in teaching a pupil to solve originals. Even if this scheme be not given directly to the pupil, it is useful to the teacher in the first place in shaping or directing the work of the pupil. It opens the way to a classification or grouping of originals in a certain progressive order. This order depends in the main on the progressively more difficult kind of auxiliary quantity or geometric tools used.

1. Our scheme requires that we first ask the pupil to solve a group of originals requiring the use only of A, the simplest and most natural auxiliary quantity, and that we avoid altogether the use of B, C and D for the time being. Experience has shown that a group of exercises in proving triangles equal best supplies what is needed.

2. We next naturally take up original exercises proved by the B method. A group of exercises in proving angles equal mainly by means of the triangle of which they are a part, some use of the isosceles triangle being made, and another group in proving line segments equal in the same way, not only gives drill in the use of the B method, but serves as a review of A.

3. Both A and B may then be reviewed, and mastered from another point of view by proving exercises concerning parallel lines. Two parallél lines and a transversal form indeed but a special kind of triangle.

4. The use of number in geometry I regard as coming partly under A, and partly under D. The use of number in that it implies the internal division of an object into equal units is A; in that is uses symbolism (the number signs) external to the object treated, is D. The use of the algebraic symbol, $x$, also something external to the object treated or represented, I regard as essentially a form of D. But it is a kind of artificial external tool not likely to be overused by the pupil. The use of numbers and of the algebraic $x$ not only may be incorporated into our scheme at this point without violence, but their use forms a good introduction to the geometric C and D. Hence at this point we have a group of exercises in proving the numerical properties of geometric figures, and also one in the proof of geometric properties by algebraic methods.

5. We are now ready for the method D, that is for the use of aggressive new and artificial geometric tools or auxiliary objects. Now that we do take up the use of these objects, we employ them systematically, and use one or more of them in every original. We begin by using first one auxiliary object (a line) in solving an original, then two, etc.

6. The use of D in the form of indirect demonstrations, and then of the locus naturally follows.

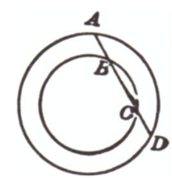
Such is the grouping of originals for Book I which is suggested by our plan. We proceed by similar graded steps in the other books of geometry.

So much for the application of aur A, B, C, D scheme in classifying originals. This much has been tested and found useful in the classroom.

The question naturally arises, can our A, B, C, D, E, X, Y method be made a help to us in other ways? There are two other ways in which I use it though rather informally. This part of the matter has not crystallized into a formal or final statement in my mind or practice.

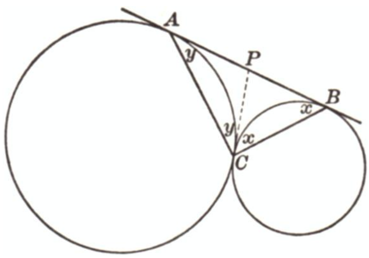
The two ways referred to are as follows:

I. The customary catalogue method of trying objects and theorems closely related to the subject matter of an original (that is, the X, Y part of the plan) may be made primary in the attention, the A, B, C, D part being merely held in the mind as a possible aid at any moment. For instance, if we have given two concentric circles with the chord $AD$ of the outer circle cutting the inner circumference at $B$ and $C$, and it is required to prove $AB = CD$, we may proceed by recalling the theorems concerning the equality of parts of a chord, and of course soon recall the theorem that the diameter of a circle perpendicular to a chord in that circle bisects the chord. If now we have our A, B, C, D scheme in mind we are in no hurry to use auxiliary quantity here, but the way being opened by the theorems recalled we have no hesitancy in drawing a diameter perpendicular to the given chord and using it as an auxiliary instrument; similarly holding our toolage schematism constantly in mind, it is natural or even inevitable for us to use at this point the axiom that if equals be subtracted from equals the remainders are equal.




II. The A, B, C, D scheme may be given the leading place and the catalogue (or X, Y) method used as something auxiliary to it. Let me illustrate this by a treatment which has occurred to me of a familiar original. We have given two circles tangent externally at $C$, and $AB$ a common external tangent of the two circles, and it required to prove that the angle $ACB$ is a right angle. If we follow the catalogue method of investigating a proof we may recall the different theorems which give methods of proving an angle a right angle and may recall that an angle inscribed in a semicircle is a right angle. If we also recall that if the common internal tangent be drawn meeting $AB$ at $P$, $PA = PC = PB$, it is easy to dray a semicircumfer-

ence with $P$ as a center and $PA$ as a radius and prove that $ACB$ is a right angle. This is a rather difficult method of proof to the average pupil, and carrying as I did my A, B, C, D scheme in mind, it occurred to me that the above method of proof is essentially the D method (the semicircumference drawn through $A$, $C$, $B$ being external to the essential part of the figure, the triangle $ACB$) and it also occurred to me, since the C method seems more natural and less artificial to the pupil, to inquire whether the difficulty of the situation could not be relieved by getting a C method of proof. This comes very simply from the fact that in the $\triangle\, ACB$




$$2x + 2y = 180°,$$

$$\therefore \quad x + y = 90°.$$

This plan of keeping our A, B, C, D scheme in mind and thus devising methods of proof of each of these leading kinds for the same original not only has the advantage of relieving the difficulties of particular pupils but also of throwing a double or manifold light on a principle or relation. I also find that the realization that there probably exists an internal proof makes the pupil more ready to accept and search for the more elegant external proof.

These methods just described of holding A, B, C, D and X, Y in the mind and giving now one, now the other, the leadership are especially useful in dealing with groups of miscellaneous exercises.

Allow me now to state in some detail the actual methods which I follow in this connection in the classroom, and then conclude with some statement of the specific advantages of the scheme as a whole and the classroom results which I, at least, get.

1. I think it is a considerable aid in teaching a class in geometry to work original exercises to have the class take a preliminary course of say one hour a week for six months in geo-

metric drawing, but I do not take space to discuss this matter in detail.

2. I have the textbooks theorems arranged so that the treatment of parallel lines is postponed till the use of the triangle both in text theorems and in originals has been thoroughly mastered. There are several reasons for this arrangement, but the chief one that bears on the subject in hand is that by this plan the originals solved during the first few weeks call for only the simplest methods of proof, viz.: the A and B methods.

3. After about eight or ten recitations on text theorems, the matter of original demonstrations is introduced in the last fifteen minutes of the daily recitation in an oral way.

4. One of the best ways to make my point of view clear will be, I think, to state the particular original which I always begin with, to state the difficulties which I meet with especially with the lower half of the class, to try to point out the source of these difficulties, and to state how I meet these difficulties.

The original I present first is the one referred to on p. 124. The data being as there stated, the object is to prove the triangles $ABO$ and $BOD$ equal or congruent. I of course ask the class to state various conditions that make two triangles equal, and a list of these is written on the blackboard. I ask the class to state whether they can point out any equal angles or any equal sides in the two given triangles. Usually the first statement volunteered is that $AO = OD$, or $AB = BD$. The reason for the fact that the class begins by searching for equal sides is that angles are implicit, and not simple, definite, palpable entities like sects; also separate lines like $AO$ and $OD$ make a more palpable impression than does a side like $BO$ used twice.

If I point out to the class that no reason can be assigned justifying the statement that $AO = OD$, I am met by the reply, "But they are equal, aren't they?" That is, the pupil tries to shift the burden of proof upon me. He assumes the right to take any pair of objects as equal, provided I cannot prove that they are unequal. I break up this habit by writing each part of the hypothesis out clearly, on the blackboard, and also each step in the proof as soon as the step is made, and by asking after each statement volunteered by a pupil the two questions "Is it in the data of the proposition?" "Is it in the preceding part of the proof?" Cut off from fallacies in this way, the pupils soon

discover that $BO = BO$ by identity, $\angle ABO = \angle OBD$ because the parts of a bisected whole are equal, and $\angle AOB = \angle BOD$ because all right angles are equal and hence that the triangles are equal by the application of the familiar theorem. Simple as are these steps, the pleasure, not to say excited joy, with which class after class makes these discoveries, impresses upon me how much we continually overestimate the abilities of beginners as they enter upon this work, and how futile it is to begin this discipline with anything like elaborate systems of analysis. Of course there are individual pupils in each class who are capable of beginning with more difficult work than is suggested above, but they seem amply repaid in watching the slower progress of the lower half of the class (1) by the varied light which they get on the subject and (2) by the stimulus which comes from a realization of their superior ability. As for the instructor, the teacher's life affords no keener pleasure than the sight of minds, hitherto dull and inert, thus germinating with the beginnings of original power.

5. At this point the study of text theorems is dropped for the time being, and five or six lessons are spent entirely in proving original exercises, viz.: exercises in proving triangles equal, and in proving lines and angles equal by means of triangles.

6. After resuming the study of text theorems, more or less time is spent on originals each day in connection with the text theorems.

7. At the end of Book I two or three weeks are spent on originals alone.

8. The original exercises studied in connection with each book are treated in the same general way.

Instruction in the use of elaborate systems of analysis is taken up gradually and largely in connection with or in explanation of particular originals.

I should state at this point that in teaching the construction problem originals as the end of Book II and in connection with subsequent books, I follow a method which affords an important economy of time, and seems to have other advantages. After the pupil has written out a full and clear statement of the solution of a few problems, I have pupils make only the drawings required (leaving all small arcs and construction lines on the drawing so as to make clear that they understand what they

are doing) and to omit all verbal statements. I tell the pupil, however, to hold himself ready to make a complete statement of the solution at any time, and occasionally require him to write out such a statement. The advantages of this method of teaching original construction problems are that (1) the pupil is able to cover much more ground, (2) he prefers using drawing instruments to writing out verbal statements, and since he takes more pleasure in his work, will work more zealously and efficiently, (3) having his mind concentrated on one line of work, he succeeds in solving much more difficult problems that he is otherwise able to solve. In the last fifteen or twenty minutes of a recitation I frequently have the main part of a class make this kind of construction solution of four problems such as the following: given the sum of the legs of a right triangle and the hypotenuse, to construct a right triangle.

Returning to our A, B, C, D, E, X, Y method as a whole, with respect to the classroom results obtained by its use, I may say that we require a pupil who is to be certified as having passed the subject of plane geometry, to pass a test on the originals of each book, and also one on the originals of plane geometry as a whole. In these pass examinations the exercises given are limited to those of medium difficulty such as proving that the diagonals of a rectangle are equal, that the angles at the base of an isosceles triangle are equal, or that the apothem of a regular inscribed triangle equals half the radius. But we try so to teach the originals required of all, that the training and culture obtained by our method shall equal that formerly obtained by the few pupils who succeeded in mastering originals by other methods.

I will mention some of these cultural results in the statement which I will now make of what I think are the advantages of my plan as a whole. This statement will conclude what I have to say.

1. The classification of original exercises in groups, each group to be solved according to a general method or principle, encourages the pupil on the one hand by limiting the amount of work immediately before him. On the other hand, the sense that he is solving in any given case not merely one original but is mastering a method arouses his interest and stimulates him by giving him a sense of achievement.

2. The order of groups of originals given above, on the one hand has the positive advantage of proceeding from the simple to the complex. On the other hand, it has the equally important negative advantage of preventing the pupil from forming the habit when in difficulty of drawing a maze of auxiliary lines which only entangle and bewilder him. Most of all, it does this for the weak pupil, who is particularly likely to fall into this error, and who especially needs this safeguard.

3. The A, B, C, D, E, X, Y scheme enables us to simplify an analysis often by cutting out or eliminating a whole group of toolage objects at once.

Thus, methods A and C apply only to relatively complex data. For instance, there is no use in trying the A method or theorems requiring it when we are trying to prove the angles at the base of an isosceles triangle equal; for these angles have no internal divisions. The same remark applies to the proof required in connection with the concentric circles mentioned on p. 129.

4. It aids the memory in recalling proofs.

5. It gives some sort of a clue or start (it may be only a rude one) for every original. It ends the cry, "I don't know how to begin"; "I can't even make a start." The pupil can always begin by noting the geometric objects given (X); and the theorems relating to these or combinations of them (Y); and then proceeding according to the A, B, C, D scale.

6. It includes all kinds of proof in the pupil's logical repertory on an equal footing in proportion to their efficiency.

I had hoped to take up the discussion of methods of proof used in text theorems of geometry, and show that under our scheme each has a certain naturalness or even inevitableness, but must omit this owing to lack of time.

7. It helps coördinate the study of geometry with other studies and that in a vital way. (1) For instance, algebra is introduced, not merely because it is a help in attaining results but also because of an important similarity of toolage nature. (Thus the use of algebra symbols is a kind of D as pointed out above.) (2) The A, B, C, D classification of auxiliary quantity applies to auxiliary quantity as used in any other department of study as physics, chemistry, psychology, language or ethics. When it is properly grasped, skill acquired in its use

in any one department should be available in every other department.

8. Hence it should be a step toward giving geometry a wider, deeper, and more definite cultural value.

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# THE SYLLABUS METHOD OF TEACHING PLANE GEOMETRY.*

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In estimating the values of different methods of teaching a subject, it is first necessary to consider the purposes for which the subject under discussion is taught. Geometry has a double purpose: its *facts* are of use in many vocations, and indeed, in occasional instances, in the everyday home life of man or woman; while, on the other hand, the *methods* and *habits* of correct geometrical work—the logical reasoning from known premises to their infallible conclusions—are of inestimable value to every intelligent person, even aside from their direct application to certain of the professions, such as branches of engineering. You will notice that I have made a distinction between the use of the *facts* of geometry in such a vocation as carpentry, metal working, and other trades, and the use of its methods in the scientific professions.

If there were only the facts of geometry to consider, a man might need to know that the sum of the angles of a triangle was a straight angle, but the reasons why would be of no concern to him. He might, then, carry a convenient table of geometrical facts in his pocket, and refer to it when necessary, or even memorize the comparatively short list of propositions which are of so-called "practical" nature. Why should he waste his time studying theoretical geometry? It would be nonsense.

It is only in case geometry has an important purpose, aside from the direct applicability of its facts to vocations, that the study of the proofs of geometrical propositions should have a place in our educational system. Has it such a use? I believe that it has, and that its great value is in developing the power

* Read at a meeting of New York Section.